MATHEMATICS LECTURE NOTE SERIES

Algebraic Topology

A First Course

REVISED

Marvin J. Greenberg
John R. Harper

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

ABP

ALGEBRAIC TOPOLOGY
A First Course

MATHEMATICS LECTURE NOTE SERIES

Volumes of the Series published from 1961 to 1973 are not officially numbered. The parenthetical numbers shown are designed to aid librarians and bibliographers to check the completeness of their holdings.

ISBN			
0-8053-5801-3	(1)	S. Lang	Algebraic Functions, 1965
0-8053-8703-X	(2)	J. Serre	Lie Algebras and Lie Groups, 1965 (4th printing, 1981)
0-8053-2327-9	(3)	P. J. Cohen	Set Theory and the Continuum Hypothesis, 1966 (5th printing, 1980)
0-8053-5808-0 0-8053-5809-9	(4)	S. Lang	Rapport sur la cohomologie des groupes, 1966
0-8053-8750-1 0-8053-8751-X	(5)	J. Serre	Algèbres de Lie semi-simples complexes, 1966
0-8053-0290-5 0-8053-0291-3	(6)	E. Artin and J. Tate	Class Field Theory, 1967 (2nd printing, 1974)
0-8053-0300-6 0-8053-0301-4	(7)	M. F. Atiyah	K-Theory, 1967
0-8053-2434-8 0-8053-2435-6	(8)	W. Feit	Characters of Finite Groups, 1967
0-8053-3555-4	(9)	Marvin J. Greenberg	Lectures on Algebraic Topology, 1966 (6th printing, with corrections, 1979)
0-8053-3757-1	(10)	Robin Hartshorne	Foundations of Projective Geometry, 1967 (4th printing, 1980)
0-0853-0660-9	(11)	H. Bass	Algebraic K-Theory, 1968
0-8053-0668-4	(12)	M. Berger and M. Berger	Perspectives in Nonlinearity: An Introduction to Nonlinear Analysis, 1968
0-8053-5208-2 0-8053-5209-0	(13)	I. Kaplansky	Rings of Operators, 1968
0-8053-6690-3 0-8053-6691-1	(14)	I. G. MacDonald	Algebraic Geometry: Introduction to Schemes, 1968
0-8053-6698-9 0-8053-6699-7	(15)	G. W. Mackey	Induced Representation of Groups and Quantum Mechanics, 1968
0-8053-7710-7 0-8053-7711-5	(16)	R. S. Palais	Foundations of Global Nonlinear Analysis, 1968
0-8053-7818-9 0-8053-7819-7	(17)	D. Passman	Permutation Groups, 1968
0-8053-8725-0	(18)	J. Serre	Abelian *l*-Adic Representations and Elliptic Curves, 1968
0-8053-0116-X	(19)	J. F. Adams	Lectures on Lie Groups, 1969
0-8053-0550-5	(20)	J. Barshay	Topics in Ring Theory, 1969
0-8053-1021-5	(21)	A. Borel	Linear Algebraic Groups, 1969
0-8053-1050-9	(22)	R. Bott	Lectures on K(X), 1969
0-8053-1430-X 0-8053-1431-8	(23)	A. Browder	Introduction to Function Algebras, 1969
		G. Choquet	Lectures on Analysis
0-8053-6955-4	(24)		Volume I. Integration and Topological Vector Spaces, 1969 (4th printing, 1980)
0-8053-6957-0	(25)		Volume II. Representation Theory, 1969 (4th printing, 1981)
0-8053-6959-7	(26)		Volume III. Infinite Dimensional Measures and Problem Solutions, 1969 (4th printing, 1981)
0-8053-2366-X 0-8053-2367-8	(27)	E. Dyer	Cohomology Theories, 1969

ALGEBRAIC TOPOLOGY

A First Course

Marvin J. Greenberg
University of California
Santa Cruz, California

John R. Harper
University of Rochester
Rochester, New York

The Advanced Book Program

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

First published 1981 by Westview Press

Published 2018 by CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

CRC Press is an imprint of the Taylor & Francis Group, an informa business

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

ISBN 13: 978-0-8053-3557-6 (pbk)

Library of Congress Cataloging in Publication Data

Greenberg, Marvin J.
Algebraic topology.

(Mathematics lecture note series ; 58)
"A revision of the first author's Lectures on algebraic topology"—P.
Bibliography: p.
Includes index.
1. Algebraic topology. I. Harper, John R., 1941- . II. Title. III. Series.
QA612.G7 514'.2 81-17108
ISBN 0-8053-3558-7 AACR2
ISBN 0-8053-3557-9 (pbk.)

MATHEMATICS LECTURE NOTE SERIES *(continued)*

ISBN			
0-8053-2420-8 0-8053-2421-6	(28)	R. Ellis	Lectures on Topological Dynamics, 1969
0-8053-2570-0 0-8053-2571-9	(29)	J. Fogarty	Invariant Theory, 1969
0-8053-3080-1 0-8053-3081-X	(30)	William Fulton	Algebraic Curves: An Introduction to Algebraic Geometry, 1969 (5th printing, with corrections, 1978)
0-8053-3552-8 0-8053-3553-6	(31)	M. J. Greenberg	Lectures on Forms in Many Variables. 1969
0-8053-3940-X 0-8053-3941-8	(32)	R. Hermann	Fourier Analysis on Groups and Partial Wave Analysis, 1969
0-8053-4551-5	(33)	J. F. P. Hudson	Piecewise Linear Topology, 1969
0-8053-5212-0 0-8053-5213-9	(34)	K. M. Kapp and H. Schneider	Completely O-Simple Semi-groups: An Abstract Treatment of the Lattice of Congruences, 1969
0-8053-5240-6 0-8053-5241-4	(35)	J. B. Keller and S. Antman (eds.)	Bifurcation Theory and Nonlinear Eigenvalue Problems, 1969
		O. Loos	Symmetric Spaces
0-8053-6620-2 0-8053-6621-0	(36)		Volume I. General Theory, 1969
0-8053-6622-9 0-8053-6623-7	(37)		Volume II. Compact Spaces and Classification, 1969
0-8053-7024-2 0-8053-7025-0	(38)	H. Matsumura	Commutative Algebra, 1970 (2nd Edition—cf. Vol. 56)
0-8053-7574-0 0-8053-7575-9	(39)	A. Ogg	Modular Forms and Dirichlet Series, 1969
0-8053-7812-X 0-8053-7813-8	(40)	W. Parry	Entropy and Generators in Ergodic Theory, 1969
0-8053-8350-6 0-8053-8351-4	(41)	W. Rudin	Function Theory in Polydiscs, 1969
0-8053-9100-2 0-8053-9101-0	(42)	S. Sternberg	Celestial Mechanics, Part I, 1969
0-8053-9102-9	(43)	S. Sternberg	Celestial Mechanics, Part II, 1969
0-8053-9254-8 0-8053-9255-6	(44)	M. E. Sweedler	Hopf Algebras, 1969
0-8053-3946-9 0-8053-3947-7	(45)	R. Hermann	Lectures in Mathematical Physics Volume I, 1970
0-8053-3942-6	(46)	R. Hermann	Lie Algebras and Quantum Mechanics, 1970
0-8053-8364-6 0-8053-8365-4	(47)	D. L. Russell	Optimization Theory, 1970
0-8053-7080-3 0-8053-7081-1	(48)	R. K. Miller	Nonlinear Volterra Integral Equations, 1971
0-8053-1875-5 0-8053-1876-3	(49)	J. L. Challifour	Generalized Functions and Fourier Analysis, 1972
0-8053-3952-3	(50)	R. Hermann	Lectures in Mathematical Physics Volume II, 1972
0-8053-2342-2 0-8053-2343-0	(51)	I. Kra	Automorphic Forms and Kleinian Groups, 1972
0-8053-8380-8 0-8053-8381-6	(52)	G. E. Sacks	Saturated Model Theory, 1972
0-8053-3103-4	(53)	A. M. Garsia	Martingale Inequalities: Seminar Notes on Recent Progress, 1973
0-8053-5664-3 0-8053-5666-5	(54)	T. Y. Lam	The Algebraic Theory of Quadratic Forms, 1973 (2nd printing, with revisions, 1980)
0-8053-6702-0 0-8053-6703-9	55	George W. Mackey	Unitary Group Representations in Physics, Probability, and Number Theory, 1978

MATHEMATICS LECTURE NOTE SERIES *(continued)*

ISBN			
0-8053-7026-9	56	Hideyuki Matsumura	Commutative Algebra, Second Edition, 1980 (2nd printing, 1981)
0-8053-0360-X	57	Richard Bellman	Analytic Number Theory: An Introduction, 1980
0-8053-3558-7 0-8053-3557-9	58	M. J. Greenberg and J. Harper	Algebraic Topology: A First Course, 1981

Other volumes in preparation

CONTENTS

PREFACE

Algebraic Topology is one of the major creations of twentieth-century mathematics. Its influence on other parts of mathematics, such as algebra [38], number theory [4, 49], algebraic geometry [27, 31, 50], differential geometry [26], and analysis [12, 1963–64] has been enormous. In its own right, it is a major tool for the investigation of topological spaces, especially manifolds. Its key idea is to attach algebraic structures to topological spaces and their maps in such a way that the algebra is both invariant under a variety of deformations of spaces and maps, and computable.

This book is intended as a first course, sufficiently comprehensive to enable the student either to use the subject in other fields of endeavor and/or to pursue its development and applications in more advanced texts and the literature.

Our presentation is a revision of the first author's *Lectures on Algebraic Topology*. The intent in revising was to make those additions of theory, examples, and exercises which updated, enhanced, and simplified the original exposition. The point of view and organizational principles of the earlier book have been maintained. Virtually all of the original book has been reproduced.

In the additional material, special attention has been given to calculations, with more geometry to balance all the algebra.

There are essentially four parts to this work: Sections 1–7 form Part I, elementary homotopy theory. Homotopy of paths and maps is defined, and the fundamental group is constructed. The classification of covering spaces by means of subgroups of the fundamental group is given, and, finally, the higher homotopy groups are defined inductively using loop spaces, following Hurewicz.

Sections 8–21, Part II, treat singular homology theory. This Part has been influenced by the lucid notes of E. Artin [3] and the work of Eilenberg-Steenrod [23]. The advantages of singular over simplicial homology theory are that, first, it applies to arbitrary topological spaces; second, it is obviously topologically invariant; third, once the excision theorem is proved, there is never again any need to subdivide, and, finally, it is easier to calculate once the basic formulas (19.16–19.18) have been proved. Combinatorial techniques are still very important in algebraic topology [36, 62, 70]. However, it is now recognized that algebraic topology encompasses at least three different categories—topological, differential, and piecewise linear. In this book we treat primarily the first (references for the second are [15, 17, 41–44, 51, 55, 68, 71]). The classical applications of homology theory to spheres are given in Sections 15, 16, and 18.

Sections 22–28 form Part III, the orientability and duality properties of manifolds. This part has been greatly influenced by notes of Dold, Puppe, and Milnor. No assumption of triangulability is needed in this treatment. The correct cohomology theory for the duality is that of Alexander-Cech; however, for brevity's sake, we only describe the Alexander-Cech cohomology module of a subspace A as the inductive limit over the neighborhoods of A of the singular cohomology modules. We show that this coincides with the singular cohomology module when A is a compact ANR.

Finally in Part IV we develop the basic features of the theory of products in cohomology. The applications include the Lefschetz fixed point theorem for compact oriented manifolds and an introduction to intersection theory in closed manifolds.

Each part is divided into several sections. These are the organizational units of the text. There is considerable flexibility (especially in the latter parts) in the order in which they may be studied. In Part II, many sections conclude with material which may be skimmed or skipped at first reading.

Most sections end with sets of exercises. No theoretical development depends on an exercise nor is further theoretical material given as exercises. Most exercises concern calculation and, as the subject develops, geometric applications are made. There are many cross-references among exercises. Refinements of calculations available with developments of the theory are offered. Similarly, improvements in geometric results are made in several sections. This process imitates the way the subject actually developed, and may help motivate the successive layers of abstraction through which the subject passes. Some exercises are accompanied with suggestions for their solution. These suggestions should not be taken too seriously. Most problems can be solved in different ways, and one's favorite solution may not receive widespread approval. But it is discouraging to be totally "stuck" so suggestions are offered to alleviate that condition.

Prerequisites for this book, besides the usual "mathematical maturity," are very few. In algebra, familiarity with groups, rings, modules, and their homomorphisms is required. From Section 20 on, some basic results for modules over principal ideal domains will be used. Only in Sections 29 and 30 is knowledge of the basic properties of the tensor product of two modules needed. The language of categories and functors is used throughout the book, although no theorems about categories are required. For all of this material, see Lang [35].

In point-set topology, the reader is presumed to be familiar with the basic facts about continuity, compactness, connectedness and pathwise-connectedness, product spaces, and quotient spaces. Only in the appendix to Section 26* do we require a nontrivial result, Tietze's extension theorem. Section 7 uses some elementary results about the compact-open topology on function spaces. For this material, see Dugundji [20] or Kelley [34].

I recommend the survey articles [44a, 62, and 75, pp. 227–31 and its bibliography] to the reader seeking further information on the extraordinary achievements in algebraic topology in recent years.

I thank M. Artin, H. Edwards, S. Lang, B. Mazur, V. Poenaru, H. Rosenberg, E. Spanier, and A. Vasquez; also my students Berkovits, Perry, and Webber, for helpful comments.

We are grateful to a number of people for helpful remarks concerning the revision. The comments of D. Anderson, E. Bishop, G. Carlsson, M. Freedman, T. Frankel, J. Lin, and K. Millett were helpful in deciding what to include and what to leave out. As the work developed, valuable remarks were made by M. Cohen, A. Liulevicius, R. Livesay, S. Lubkin, H. Miller, R. Mandelbaum, N. Stein, and A. Zabrodsky.

The typing of the manuscript was expertly done by S. Agostinelli, R. Colon, and M. Lind. Additional figures were drawn by D. McCumber.

Special thanks are extended to Doris, Jennifer, and Allison for not overreacting to neglect endured during preparation and assembly of this material.

Lastly, we thank Errett Bishop for suggesting that we collaborate on this book.

MARVIN J. GREENBERG
JOHN R. HARPER

*(26.17)

ALGEBRAIC TOPOLOGY
A First Course

Part 1
ELEMENTARY HOMOTOPY THEORY

Introduction to Part I

The wellspring of ideas leading to algebraic topology was the perception, developed largely in the latter half of the nineteenth century, that many properties of functions were invariant under "deformations." For example, Cauchy's theorem and the calculus of residues in complex analysis assert invariance of complex integrals with respect to continuous deformations of curves. Perhaps the true starting point was Riemann's theory of abelian integrals. It was here that the significance of the connectivity of surfaces was recognized. The interested reader is strongly encouraged to examine Felix Klein's exposition of Riemann's theory [80], during the study of algebraic topology.

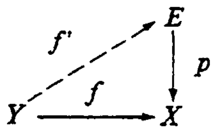
It was Poincaré who first systematically attacked the problem of attaching numerical topological invariants to spaces. In his investigations, he perceived the difference between curves *deformable* to one another and curves *bounding* a larger space. The former idea led to the introduction of homotopy and the fundamental group, while the latter led to homology.

The development of these ideas into a mathematical theory is elaborate. However, the idea guiding the development is easily described. Certain functors are constructed. Thus to each topological space X is assigned a group $F(X)$, and to each map $f: X \to Y$ (a "map" of topological spaces will always mean a "continuous map" unless otherwise stated) is assigned a homomorphism $F(f): F(X) \to F(Y)$ such that

(1) If $Y = X$ and $f =$ identity, then $F(f) =$ identity,

(2) If $g: Y \to Z$, then $F(gf) = F(g)F(f)$.

Illustration: Suppose we have a diagram of topological spaces and maps



and the problem is to find f' such that $pf' = f$. Applying the functor F we see that a necessary condition for a solution to exist is that $F(f)$ send $F(Y)$ into the subgroup $F(p)(F(E))$ of $F(X)$. In certain cases later we will see this is also sufficient (6.1).

Illustration: Suppose $f : X \to Y$ is a homeomorphism. Then by functoriality $F(f^{-1})$ is inverse to $F(f)$, so that $F(f)$ is an isomorphism. Thus a *necessary* condition (but usually not sufficient) that X and Y be homeomorphic is that $F(X)$ and $F(Y)$ be isomorphic groups. This is usually the easiest way to prove that two given spaces with similar topological properties are not homeomorphic.

Illustration: Suppose $i : A \to X$ is the inclusion map of a subspace A into X and our problem is to find a map $r : X \to A$ such that ri is the identity map of A (such a map r is called a *retraction* of X onto A). By functoriality, $F(r)F(i)$ equals the identity transformation of $F(A)$, so that $F(i)$ sends $F(A)$ isomorphically onto a subgroup of $F(X)$. If we happen to know, e.g., that $F(X)$ is trivial while $F(A)$ is not, it then follows that no retraction can exist. This is the way the Brouwer Fixed Point Theorem is proved (4.11 and 15.7).

The reader may construct some more illustrations to convince himself of the fruitfulness of this point of view.

1. Arrangement of Part I

In Part I, we treat the fundamental group and the closely related notion of covering space. The geometric idea for the construction of the fundamental group functor is homotopy of paths. Roughly speaking, a homotopy of a path is a deformation leaving the end points fixed. A composition of paths may be defined when the end point of one agrees with the initial point of the other. Familiar algebraic properties, like associativity, do not hold, but do hold up to homotopy. The result is a group structure on equivalences classes, called the fundamental group. This group is not just a topological invariant, but invariant under a larger class of maps, called *homotopy equivalences*. These topics are treated in Sections 2 and 3.

In order to exploit the fundamental group, we must be able to calculate it. There are two principal routes to calculation: the Seifert-Van Kampen theorem and the use of covering spaces. The versions of the former used in this text are stated in (4.12). There are several excellent accounts available in other texts, so we do not reproduce the details. Our treatment of the fundamental group of the circle is the prototype for the theory of covering spaces. The lifting theorem for covering spaces (6.1), besides being useful, is an outstanding example of the blend of algebra and geometry that gives this subject its special flavor. Part I concludes with a brief discussion of higher homotopy groups, introduced by means of loop spaces.

2. Homotopy of Paths

Consider, in the plane, the problem of integrating a function f of a complex variable around a closed curve C, e.g., the unit circle. We have, for example,

$$\int_C z\, dz = 0$$

$$\int_C \frac{dz}{z} \neq 0$$

What is the difference? We take the point of view that C can be "shrunk to a point" within the domain of analyticity of z (i.e., the whole plane), hence integrating around C is equivalent to integrating at a point, which gives 0. On the contrary C cannot be "shrunk to a point" within the domain of $1/z$.

More precisely, let σ, τ be *paths* in a space X (i.e., maps of the unit interval I into X) with the same end points (i.e., $\sigma(0) = \tau(0) = x_0$, $\sigma(1) = \tau(1) = x_1$). We say *$\sigma$ and τ are homotopic with end points held fixed* written

$$\sigma \simeq \tau \text{ rel } (0, 1)$$

if there is a map $F : I \times I \to X$ such that

(1) $F(s, 0) = \sigma(s)$ all s

(2) $F(s, 1) = \tau(s)$ all s

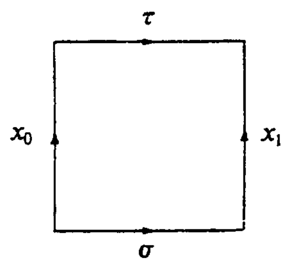
(3) $F(0, t) = x_0$ all t

(4) $F(1, t) = x_1$ all t

F is called a *homotopy* from σ to τ. For each t, $s \to F(s, t)$ is a path F_t from x_0 to x_1, and $F_0 = \sigma$, $F_1 = \tau$. We write

$$F_t : \sigma \simeq \tau \qquad \text{rel}\,(0, 1)$$

Pictorially:



In particular if σ is a *loop* at x_0 (i.e., $x_1 = x_0$) and τ is the constant loop $\tau(s) = x_0$ for all s, and if $\sigma \simeq \tau$ rel (0, 1), we say that "σ can be shrunk to a point," or is *homotopically trivial.*

Then the correct statement of Cauchy's Theorem is that $\int_C f(z)dz = 0$ for all loops C in the domain X of analyticity of f which are homotopically trivial (more generally, homologically trivial).

The following properties of relation $\simeq$ are easily proved:

(1) $\sigma \simeq \sigma \quad \text{rel}\,(0, 1)$

(2) $\sigma \simeq \tau \quad \text{rel}\,(0, 1) \Rightarrow \tau \simeq \sigma \ \text{rel}\,(0, 1)$

(3) $\sigma \simeq \tau \quad \text{rel}\,(0, 1)$ and $\tau \simeq \rho \ \text{rel}\,(0, 1) \Rightarrow \sigma \simeq \rho \ \text{rel}\,(0, 1)$

Thus we can consider the homotopy classes $[\sigma]$ of paths σ from x_0 to x_1 under the equivalence relation $\simeq$.

If σ is a path from x_0 to x_1 and τ is now taken to be a path from x_1 to x_2, we define a path $\sigma\tau$ from x_0 to x_2 by first travelling along σ, then along τ; more precisely we set

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & 0 \le t \le 1/2 \\ \tau(2t - 1) & 1/2 \le t \le 1 \end{cases}$$

(4) $\sigma \simeq \sigma'$ rel (0, 1) and $\tau \simeq \tau'$ rel (0, 1) $\Rightarrow \sigma\tau \simeq \sigma'\tau'$ rel (0, 1).

Proof: If $F_t : \sigma \simeq \sigma'$ rel (0, 1), $G_t : \tau \simeq \tau'$ rel (0, 1), then

$$F_t G_t : \sigma\tau \simeq \sigma'\tau' \text{ rel } (0, 1). \quad \blacksquare$$

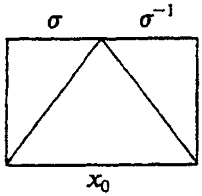
Thus we can multiply the *class* of σ on the right by the *class* of τ without ambiguity, always supposing the end point of σ equals the initial point of τ.

(2.1) *Theorem. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X at x_0. If multiplication in $\pi_1(X, x_0)$ is defined as above, $\pi_1(X, x_0)$ becomes a group, in which the neutral element is the class of the constant loop at x_0 and the inverse of a class $[\sigma]$ is the class of the loop σ^{-1} defined by*

$$\sigma^{-1}(t) = \sigma(1 - t) \qquad 0 \leq t \leq 1$$

(i.e., travel backwards along σ).

Proof: We will prove that $\sigma\sigma^{-1} \simeq x_0$, where now x_0 denotes also the constant loop at the point x_0. The homotopy is given by the following diagram:



Thus, we define $F(s, t)$ by

$$F(s, t) = \begin{cases} \sigma(2s) & 0 \leq 2s \leq t \\ \sigma(t) & t \leq 2s \leq 2 - t \\ \sigma^{-1}(2s - 1) & 2 - t \leq 2s \leq 2 \end{cases}$$

Clearly these functions are continuous on each triangle and they agree on the intersections, hence by an elementary argument F is continuous on the whole square.

The proof that multiplication is associative (up to homotopy) can be done similarly, as can the proof that the class of x_0 is the neutral element.

$$\text{Define } F(s,t) = \begin{cases} \sigma\left(\dfrac{4s}{t+1}\right) & 0 \le s \le \frac{1}{4}(t+1) \\ \tau(4s-t-1) & \frac{1}{4}(t+1) \le s \le \frac{1}{4}(t+2) \\ \omega\left(\dfrac{4s-t-2}{2-t}\right) & \frac{1}{4}(t+2) \le s \le 1 \end{cases}$$

to establish $(\sigma\tau)\omega \simeq \sigma(\tau\omega)$ rel (0, 1).

$$\text{Define } F(s,t) = \begin{cases} \sigma\left(\dfrac{2s}{t+1}\right) & 0 \le s \le \dfrac{t+1}{2} \\ x_0 & \dfrac{t+1}{2} \le s \le 1 \end{cases}$$

to establish that the constant path at x_0 is the neutral element of the fundamental group. ∎

Is there a relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$? There certainly is not if x_0 and x_1 lie in different path-connected components of X. However, we have the following result.

(2.2) *Proposition*. Let α *be a path from* x_0 *to* x_1. *The mapping* $[\sigma] \to [\alpha^{-1}\sigma\alpha]$ *is an isomorphism* α_* *of the group* $\pi_1(X, x_0)$ *onto* $\pi_1(X, x_1)$.

Proof: It is clearly a homomorphism, and $(\alpha^{-1})_*$ is its inverse (where α^{-1} is the path defined as in 2.1). ∎

(2.3) *Corollary. If* X *is pathwise connected, the group* $\pi_1(X, x_0)$ *is independent of the point* x_0, *up to isomorphism.*

In that case we often write simply $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it *the fundamental group* of X.

We would like π_1 to be a functor from spaces to groups, but since $\pi_1(X, x_0)$ does depend on the base point x_0 in the general case, we must put the base points into our category if we are to obtain a functor. So define the category of *pointed topological spaces* to have as objects pairs (X, x_0), and as morphisms the maps $f: X \to Y$ such that $f(x_0) = y_0$. For any such f we obtain an induced homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by $f_*[s] = [f \circ s]$. One verifies easily that this is well-defined and a homomorphism. Moreover,

(1) $Y = X$ and $f =$ identify $\Rightarrow f_* =$ identity;

(2) Given $g : (Y, y_0) \to (Z, z_0)$, $(gf)_* = g_* f_*$.

Thus we can speak of *the fundamental group functor* from the category of pointed topological spaces to the category of groups.

3. Homotopy of Maps

Since paths are maps of I into X, we can try to replace I by any space Y and define homotopy. Thus we no longer have end points but we can substitute a subspace $A \subset Y$ for the set $\{0, 1\}$.

Given maps $f, g : Y \to X$ such that $f \mid A = g \mid A$, we say

$$f \simeq g \quad \text{rel } A$$

if there is a map $F : Y \times I \to X$ satisfying

(1) $F(y, 0) = f(y)$ all $y \in Y$

(2) $F(y, 1) = g(y)$ all $y \in Y$

(3) $F(y, t) = f(y) = g(y)$ all $y \in A, t \in I$

In case A is empty, we write simply

$$f \simeq g$$

Once again we obtain an equivalence relation.

Example 1: Let $X = Y = \mathbf{R}^n$, let f be the identity, g the constant map 0. Then

$$F(x, t) = t\,x$$

defines a homotopy from g to f.

If X is a space such that the identity map on X is homotopic to a constant map on some point in X, we say X is *contractible.*

(3.1) *Exercise.* X is contractible if and only if for any space Y any two maps of Y into X are homotopic. A contractible space is pathwise connected.

Example 2: Every convex subset X of Euclidean space is contractible. For if $f_1, f_2 : Y \to X$, we define a homotopy by

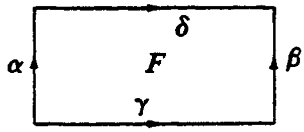
$$F(y, t) = t f_1(y) + (1 - t) f_2(y) \quad y \in Y, t \in I$$

Call a space *simply connected* if it is pathwise connected and its fundamental group is trivial.

(3.2) *Proposition. A contractible space is simply connected.*

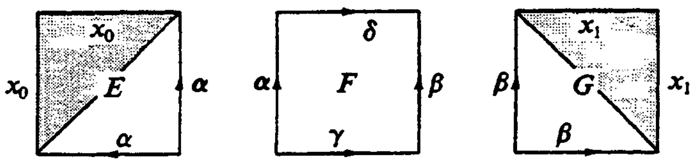
Proof: This is not entirely obvious, because although every loop σ at a point x_0 is homotopic as a map with the constant loop, we do not know they are homotopic *relative to* (0, 1).

(3.3) *Lemma. Given* $F : I \times I \to X$. *Set* $\alpha(t) = F(0, t)$; $\beta(t) = F(1, t)$, $\gamma(s) = F(s, 0)$, $\delta(s) = F(s, 1)$, *so that diagrammatically*



Then $\delta \simeq \alpha^{-1}\gamma\beta \quad$ rel (0, 1).

Proof: The proof is by juxtaposing 3 squares



where $x_0 = \delta(0)$, $x_1 = \delta(1)$, and

$$E(s, t) = \begin{cases} x_0 & s \leq t \\ \alpha(1 + t - s) & s \geq t \end{cases}$$

$$G(s, t) = \begin{cases} \beta(t + s) & 1 - s \geq t \\ x_1 & 1 - s \leq t \end{cases}$$

■

If now X is contractible, we can obtain such an F with $\delta = \sigma$, $\gamma = x_0$, and $\alpha = \beta$ (since σ induces a map of the circle into X which is homotopic to the constant map at x_0), hence σ is homotopically trivial.

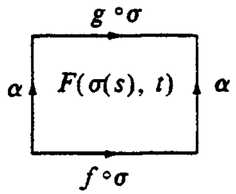
(3.4) *Corollary. Let f, g be homotopic maps $Y \to X$ by means of a homotopy $F: Y \times I \to X$. Let $y_0 \in Y$, $x_0 = f(y_0)$, $x_1 = g(y_0)$. Let α be the path from x_0 to x_1 given by*

$$\alpha(t) = F(y_0, t) \quad t \in I$$

Then we have a commutative triangle

$$\begin{array}{ccc} \pi_1(Y, y_0) & \xrightarrow{f_*} & \pi_1(X, x_0) \\ & \searrow^{g_*} & \downarrow \alpha_* \\ & & \pi_1(X, x_1) \end{array}$$

Proof: For any loop σ at y_0, we have



■

(3.5) *Corollary. Under the above conditions, f_* is an isomorphism if and only if g_* is.*

A map $f: Y \to X$ is called a *homotopy equivalence* if there is a map $f': X \to Y$ such that

$$ff' \simeq \text{identity map of } X$$

$$f'f \simeq \text{identity map of } Y$$

If such an f exists we say X and Y are *homotopically equivalent spaces*. For example, X is contractible if and only if it is homotopically equivalent to a point.

(3.6) *Corollary. If f is a homotopy equivalence then f_* is an isomorphism $\pi_1(Y, y_0) \to \pi_1(X, f(y_0))$ for all $y_0 \in Y$.*

For by the previous corollary, $f_* f'_*$ and $f'_* f_*$ are both isomorphisms. ■

Thus the fundamental group of a path-connected space is a *homotopy* *invariant* (*a fortiori*, a topological invariant). The relation of homotopy equivalence is cruder than topological equivalence. For example, the most elementary homotopy equivalence is a contraction—shrinking a portion of a space to a point. The importance of the idea of homotopy equivalence for algebraic topology lies in the fact that constructions used to attach algebra to spaces usually lead to homotopy invariant structures. Furthermore, the understanding of homotopy types is a base from which to attack more subtle questions involving topological type.

Exercise: Classify the letters of your favorite alphabet according to homotopy type and topological type.

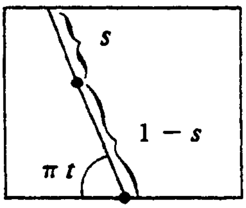
(3.7) *Exercise*. Let X be path connected. Then the following are equivalent:

(1) X is simply connected;

(2) Every map of the unit circle S^1 into X extends to a map of the closed unit disc E^2 into X;

(3) If σ, τ are paths in X with the same initial points and the same terminal points then $\sigma \simeq \tau$ rel (0, 1).

Hints: To show (1) $\Longleftrightarrow$ (2), represent E^2 as a quotient space of $I \times I$ by sending the point (s, t) $\in I \times I$ onto the point $t\,e^{2\pi i s}$. To show (1) $\Rightarrow$ (3), use the transformation of the square described by the diagram



(3.8) *Exercise*. Let $CX = X \times I/X \times \{0\}$, be the *cone* on X. Regard $X \subset CX$ via $x \to (x, 1)$. Generalize (3.7)(2) to show $f: X \to Y$ is homotopically trivial if and only if f extends to $\bar{f}: CX \to Y$.

(3.9) *Exercise*. Suppose Y is contractible to a point y. Show

$$f: X \to X \times Y$$

by $f(x) = (x, y_0)$ and projection $p: X \times Y \to X$ are homotopy equivalences.

(3.10) *Exercise*. Let $f, g: S^n \to S^n$ be maps such that for all $x \in S^n$, $f(x)$ and $g(x)$ are not antipodal. Show $f \simeq g$. If in addition there is $x_0 \in X$ such that $f(x_0) = g(x_0)$, show $f \simeq g$ rel x_0.

(3.11) *Exercise*. Suppose X and Y have the same homotopy type. Show that the arc components of X and Y are in one-to-one correspondence.

(3.12) *Exercise*. Let X be arc connected and suppose every $f: S^1 \to X$ is homotopically trivial but not necessarily by a homotopy leaving the base point fixed. Show $\pi_1(X, x_0) = 0$.

4. Fundamental Group of the Circle

We study the circle S^1 via the line **R**. It turns out that the homotopy class of a loop is determined by the number of times it "winds around," the number being negative if the "winding" is opposite to the given orientation on S^1.

More precisely, S^1 is the group of complex numbers of absolute value 1. We have a continuous homomorphism $\phi : \mathbf{R} \to S^1$ ($\mathbf{R}$ as additive group) given by

$$\phi(x) = e^{2\pi i x} \qquad x \in \mathbf{R}$$

Moreover, the mapping ϕ is an open mapping, as is easily verified. Hence ϕ maps the open interval $(-\frac{1}{2}, +\frac{1}{2})$ on the line homeomorphically onto $S^1 - \{-1\}$; let ψ be its inverse on that set. We need two key lemmas.

(4.1) *Lifting Lemma. If σ is a path in S^1 with initial point* 1, *there is a unique path σ' in* **R** *with initial point* 0 *such that $\phi \circ \sigma' = \sigma$.*

(4.2) *Covering Homotopy Lemma. If also τ is a path in S^1 with the initial point* 1 *such that*

$$F : \sigma \simeq \tau \quad \text{rel } (0, 1)$$

then there is a unique $F' : I \times I \to \mathbf{R}$ such that

$$F' : \sigma' \simeq \tau' \quad \text{rel } (0, 1)$$
$$\phi \circ F' = F$$

Proof: We prove both lemmas at the same time. Let Y be either I or $I \times I$, $f\colon Y \to S^1$ either σ or F, $0 \in Y$ either 0 or (0, 0). Since Y is compact, f is uniformly continuous, so there exists $\varepsilon > 0$ such that $|y - y'| < \varepsilon \Rightarrow |f(y) - f(y')| < 1$; in particular for such y and y', $f(y) \neq -f(y')$, so $\psi(f(y)/f(y'))$ is defined. We can find N so large that $|y| < N\varepsilon$ for all $y \in Y$. Set

$$f'(y) = \psi\left(f(y)/f\left(\frac{N-1}{N}y\right)\right) + \psi\left(f\left(\frac{N-1}{N}y\right)\Big/ f\left(\frac{N-2}{N}y\right)\right) + \cdots + \psi\left(f\left(\frac{1}{N}y\right)/f(0)\right)$$

Then f' is continuous $Y \to \mathbf{R}$, $f'(0) = 0$, and $\phi \circ f' = f$.

If we had $f''\colon Y \to \mathbf{R}$, $f''(0) = 0$, and $\phi \circ f'' = f$, then $f' - f''$ would be a (continuous) map of Y into the kernel of ϕ, i.e., into $\mathbf{Z}$. Since Y is connected, $f' - f''$ is constant, hence $f' = f''$.

In the case $Y = I \times I$, $f = F$, $f' = F'$, we see $F'\colon \sigma \simeq \tau$. In fact, the homotopy is relative to (0, 1), for on $0 \times I$, $\phi \circ F' = F = 1$, hence $F'(0 \times I) \subset \mathbf{Z}$, so by connectedness again, $F'(0 \times I) = 0$. Similarly $F'(1 \times I)$ is constant. (For another proof, see 5.1–5.3.) ■

(4.3) *Corollary. The end point of σ' depends only on the homotopy class of σ.*

Define a map $\chi\colon \pi_1(S^1, 1) \to \mathbf{Z}$ by $\chi[\sigma] = \sigma'(1)$. We have just shown χ is well defined. It is a homomorphism: Given $[\sigma]$, $[\tau] \in \pi_1(S^1, 1)$. Let $m = \sigma'(1)$, $n = \tau'(1)$. Let τ'' be the path from m to $m + n$ in $\mathbf{R}$ given by $\tau''(s) = \tau'(s) + m$. Then $\phi \circ \tau'' = \tau$ also, so $\sigma'\tau''$ is the lifting of $\sigma\tau$ with initial point 0; its end point is $m + n$. Hence $\chi([\sigma][\tau]) = \chi[\sigma] + \chi[\tau]$.

χ is onto: Given n, define $\sigma'(s) = ns$. If $\sigma = \phi \circ \sigma'$, $\chi[\sigma] = n$.

χ is a monomorphism: Suppose $\chi[\sigma] = 0$, so σ' is a loop in $\mathbf{R}$ at 0. $\mathbf{R}$ being contractible, $\sigma' \simeq 0$ rel (0, 1), whence applying ϕ, $\sigma \simeq 1$ rel (0, 1), $[\sigma] = 1$. This proves the following theorem.

(4.4) *Theorem.*

$$\pi_1(S^1) \cong \mathbf{Z}$$

Remark: The only property of S^1 used in this proof is that it is a topological group, quotient of **R** by **Z**. The only property of **R** used in this proof is that it is a simply connected topological group. The only property of **Z** used is that it is a discrete subgroup of **R**. Thus exactly the same argument gives a more general result.

(4.5) *Theorem. If G is a simply connected topological group, H a discrete normal subgroup, then*

$$\pi_1(G/H, 1) \cong H$$

There is one detail to check: We must find an open neighborhood V of 1 in G which is mapped homeomorphically onto an open neighborhood of 1 in G/H by $\phi : G \to G/H$, so that we can use ψ as before. Since H is discrete, there is an open neighborhood U of 1 such that $U \cap H = \{1\}$. By continuity of the map $(g_1, g_2) \to g_1 g_2^{-1}$, there is another open neighborhood $V \subset U$ of 1 such that $g_1, g_2 \in V \Rightarrow g_1 g_2^{-1} \in U$. This is the V we need. ■

(4.6) *Exercise*. A discrete normal subgroup of a connected topological group is central. Hence $\pi_1(G/H)$ is commutative.

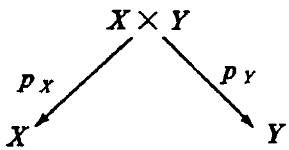
(4.7) *Corollary. The fundamental group of a torus is* $\mathbf{Z} \times \mathbf{Z}$.

For the torus is topologically $S^1 \times S^1$, hence is a topological group isomorphic to $(\mathbf{R} \times \mathbf{R})/(\mathbf{Z} \times \mathbf{Z})$. ■

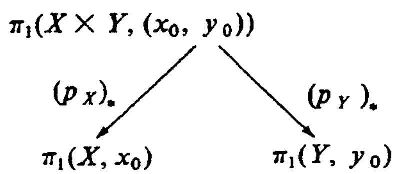
The last result could have been derived in another way

(4.8) *Proposition. Given spaces X, Y, $x_0 \in X$, $y_0 \in Y$, we have* $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof: The isomorphism is obtained as follows. Let



be the projection maps. They induce homomorphisms



hence a homomorphism $((p_X)_*, (p_Y)_*)$ of $\pi_1(X \times Y, (x_0, y_0))$ into $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. This homomorphism is an isomorphism, because it has the following inverse: Given loops σ at x_0, τ at y_0, assign to the pair $([\sigma], [\tau])$ the class of the loop (σ, τ) at (x_0, y_0) defined by

$$(\sigma, \tau)(t) = (\sigma(t), \tau(t)) \quad \text{for all} \quad t \in I$$

We leave the details as an exercise, as well as the verification that the isomorphism is *functorial* in (X, Y). ■

As an application our theorem and as an illustration of the method of algebraic topology, we prove the following theorem.

(4.9) *Theorem. The circle is not a retract of the closed unit disc.*

This means there is no map f of E^2 onto S^1 whose restriction to S^1 is the identity. Suppose we had such an f. Let $i: S^1 \to E^2$ be the inclusion map, so $fi =$ identity. Applying the fundamental group functor we get

$$\underbrace{\pi_1(S^1, 1) \xrightarrow{i_*} \pi_1(E^2, (1, 0)) \xrightarrow{f_*} \pi_1(S^1, 1)}_{\text{identity}}$$

But this means $\mathbf{Z} \to 0 \to \mathbf{Z}$ is the identity, which is impossible. ■

Note: In fact, there is no map $f: E^2 \to S^1$ such that $fi \simeq$ identity, for if there were, $f_* i_*$ would be an isomorphism (by 3.5), which is impossible.

(4.10) *Exercise.* Prove the circle is a strong deformation retract of the closed disc minus the origin. (A subspace A of a space X is a *strong deformation retract* of X if there is a homotopy $F_t: X \to X$ such that $F_0 =$ identity of X, $F_t | A =$ identity of A for all t, F_1 maps X into A.)

(4.11) *Corollary. Any continuous map of the closed disc into itself has a fixed point.*

This is the case $n = 2$ of the *Brouwer Fixed Point Theorem*, to be proved later for all n. (Exercise: Do the case $n = 1$.) Suppose $f: E^2 \to E^2$ has no fixed point. For any $x \in E^2$, join x to $f(x)$ by a line; move along this line in the direction from $f(x)$ to x until the point $r(x)$ on S^1 is reached. Then r is a retraction of E^2 on S^1, contradiction. ■

(4.12) *Exercise.* Suppose the space X is the union of two open sets U and V, such that $U \cap V$ is nonempty and pathwise connected, and U, V are each simply connected. Then X is simply connected. (This is a special case of Van